Published by Institute of Physics Publishing for SISSA/ISAS
Received: *March 1, 2002*
Accepted: *April 30, 2002*

Born-Oppenheimer corrections to the effective zero-mode hamiltonian in SYM theory

Andrei V. Smilga

SUBATECH, Université de Nantes
4 rue Alfred Kastler, BP 20722, Nantes 44307, France
E-mail: smilga@subatech.in2p3.fr

Abstract: We calculate the subleading terms in the Born-Oppenheimer expansion for the effective zero-mode hamiltonian of $\mathcal{N}=1$, $d=4$ supersymmetric Yang-Mills theory with any gauge group. The hamiltonian depends on $3r$ abelian gauge potentials A_k^s ($k=1,2,3$, $s=1,\ldots,r$, r is the rank of the group), and their superpartners. The hamiltonian belongs to the class of $\mathcal{N}=2$ supersymmetric QM hamiltonian constructed earlier by Ivanov and I. Its bosonic part describes the motion over the $3r$-dimensional manifold with a special metric. The corrections explode when the root forms $\alpha_j(A_k^s)$ vanish and the Born-Oppenheimer approximation breaks down.

Keywords: Supersymmetric Effective Theories, Field Theories in Lower Dimensions, M(atrix) Theories.

 http://jhep.sissa.it/archive/papers/jhep042002054/jhep042002054.pdf

Contents

1. Introduction

The vacuum dynamics of supersymmetric gauge theories has been a subject of intense interest and study since 1982 when Witten introduced the notion of supersymmetric index and calculated it for pure supersymmetric gauge theories with unitary and symplectic groups [1].

One of the ways to tackle the problem is to put the system in a very small spatial box and truncate all higher Fourier modes. The problem is then reduced to a pure quantum mechanical problem, which will be the starting point of our discussion here.

We hasten to comment that such a crude truncation is not quite consistent. A more correct procedure is to integrate out the higher Fourier modes in the Born-Oppenheimer spirit. This will be discussed in more details in section 3. For time being, let us consider a QM system with 2 complex supercharges obtained by dimensional reduction from the original 4-dimensional pure Yang-Mills $\mathcal{N}=1$ supersymmetric theory based on the gauge group G. This system is called sometimes supersymmetric *matrix model* and is interesting on its own.

The hamiltonian has the form

$$H = \frac{1}{2}E_i^a E_i^a + \frac{g^2}{4} f^{abe} f^{cde} A_i^a A_j^b A_i^c A_j^d - ig f^{abc} \bar{\lambda}^{a\alpha} (\sigma_i)_\alpha^{\ \beta} \lambda_\beta^b A_i^c \,,$$
$$i = 1,2,3\,, \qquad \alpha = 1,2\,, \qquad a = 1,\ldots,\dim(G)\,. \tag{1.1}$$

A_i^a are the gauge potentials, $E_i^a = -i\partial/\partial A_i^a$ are their canonical momenta operators, and λ_α^a and $\bar{\lambda}^{a\alpha} = \partial/\partial\lambda_\alpha^a$ are the fermionic gluino variables and their momenta.[1] The Hilbert space includes only the physical states annihilated by the Gauss law constraints

$$G^a \Psi = f^{abc}\left(E_i^b A_i^c + i\bar{\lambda}^{b\alpha}\lambda_\alpha^c\right)\Psi = 0\,. \tag{1.2}$$

[1]The indices are raised and lowered with the ϵ-symbol: $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta, \epsilon^{12} = -\epsilon_{12} = 1$. A hermitean conjugate of an operator A_α is by definition $\bar{A}^\alpha$. Also, $\psi\chi \equiv \epsilon^{\alpha\beta}\psi_\alpha\chi_\beta,\ \bar{\psi}\bar{\chi} \equiv \epsilon_{\alpha\beta}\bar{\psi}^\alpha\bar{\chi}^\beta$.

The system has two conserved complex supercharges

$$Q_\alpha = \frac{1}{\sqrt{2}}(\sigma_i)_\alpha{}^\beta \lambda^a_\beta \left[E^a_i - \frac{ig}{2}\epsilon_{ijk} f^{abc} A^b_j A^c_k\right] \tag{1.3}$$

(they formed a Weyl spinor before reduction) and, being restricted on the Hilbert space (1.2), enjoys the $\mathcal{N} = 2$ SQM algebra $\{\bar{Q}^\alpha, Q_\beta\}_+ = \delta^\alpha_\beta H$. The classical potential in eq. (1.1) vanishes in the "vacuum valleys" with $f^{abc}A^b_i A^c_j = 0$. Due to supersymmetry, degeneracy along the valleys survives also after quantum corrections are taken into account. As a result, the system tends to escape along the valleys, the wave function of the low-energy states is delocalized, and the spectrum is continuous (this implies, incidentally, the continuity of the mass spectrum of supermembranes [2, 3]).

The vacuum valley (or moduli space) is parametrized by r 3-dimensional vectors A^s_i lying in the Cartan subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of the gauge group G. As the motion along the valley is infinite, the characteristic values of the moduli A^s_i are large. We can subdivide the physical bosonic variables (there are altogether $3\dim(G) - \dim(G) = 2\dim(G)$ such variables) into $3r$ slow variables A^s_i and the fast variables aligned along the root vectors of $\mathfrak{g}$. It is natural then to integrate over the fast variables and to write down the effective hamiltonian depending only on the slow variables A^s_i and their superpartners λ^s_α. To leading order, this hamiltonian has a rather simple form [1, 2, 4]:[2]

$$H^{\rm eff} = \sum_{s=1}^r \frac{1}{2} E^s_i E^s_i \,. \tag{1.4}$$

The corresponding supercharges are

$$Q^{\rm eff}_\alpha = \sum_{s=1}^r \frac{1}{\sqrt{2}}(\sigma_i)_\alpha{}^\beta \lambda^s_\beta E^s_i \,. \tag{1.5}$$

This paper is devoted to calculation of the subleading corrections to eqs. (1.4), (1.5). We use the method developped earlier in ref. [5] to calculate the corrections to the effective hamiltonian in supersymmetric QED. The results are also similar. We will show that, for SU(2) theory, the effective supercharge and the hamiltonian are given by the expression

$$\begin{aligned}
Q_\alpha &= \sqrt{\frac{1}{2}}\left[(\sigma_k)_\alpha{}^\beta \psi_\beta f(\mathbf{c}) P_k + i\partial_k f(\mathbf{c}) \bar\psi \sigma_k \psi \psi_\alpha\right], \\
\bar{Q}^\alpha &= \sqrt{\frac{1}{2}}\left[\bar\psi^\beta (\sigma_k)_\beta{}^\alpha f(\mathbf{c}) P_k - i\partial_k f(\mathbf{c}) \bar\psi \sigma_k \psi \bar\psi^\alpha\right], \\
H &= \frac{1}{2} f(\mathbf{c}) P^2_k f(\mathbf{c}) - \epsilon_{jkp} \bar\psi \sigma_j \psi f(\mathbf{c}) \partial_p f(\mathbf{c}) P_k - \frac{1}{2} f(\mathbf{c}) \partial^2_k f(\mathbf{c}) (\bar\psi\psi)^2
\end{aligned} \tag{1.6}$$

with

$$f(\mathbf{c}) = 1 + \frac{3}{4g|\mathbf{c}|^3} \,. \tag{1.7}$$

Here $c_i \equiv A^3_i$ and $\psi_\alpha \equiv \lambda^3_\alpha$. The differential operators $P_k = -i\partial/\partial c_k$ and $\bar\psi^\alpha = \partial/\partial\psi_\alpha$ act on everything on the right they find.

[2] In addition, the requirement of Weyl invariance of the wave functions should be imposed, but we are not going to discuss it here.

The results (1.6) have exactly the same form as the effective supercharges and hamiltonian for the photon and photino zero modes in dimensionally reduced SQED found in ref. [5]. The only difference is that the function $f(\mathbf{c})$ is modified: for SQED, the coefficient in the second term in eq. (1.7) is $-1/4$ instead of $3/4$. We will show that for theories based on the groups of higher rank r, the effective hamiltonian is given by a generalization of eq. (1.6) involving a sum over the roots of $\mathfrak{g}$.

2. Supersymmetric QED

To make the discussion self-contained, we have to remind briefly the basic steps of the calculation of $H^{\rm eff}$ in supersymmetric QED.

The theory involves the photon A_μ, the photino ψ_α, two Weyl fermions with opposite charges ξ_α, η_α and two oppositely charged scalars φ, χ. The charged fields are assumed massless. We assume also that there is no spatial dependence and we have quantum mechanics rather than field theory. The dynamical variables A_i are slow and the variables φ, χ are fast. The supercharges and the hamiltonian of the system are convenient to represent as

$$Q_\alpha = Q_\alpha^{(0)} + Q_\alpha^{(1)}\,, \qquad H = H^{(0)} + H^{(1)} + H^{(2)}\,, \tag{2.1}$$

where[3]

$$\begin{aligned} Q_\alpha^{(0)} &= \left[-\pi_\varphi \delta_\alpha^\beta + ie\bar\varphi A_k (\sigma_k)_\alpha^{\ \beta}\right]\xi_\beta + \left[-\pi_\chi \delta_\alpha^\beta - ie\bar\chi A_k (\sigma_k)_\alpha^{\ \beta}\right]\eta_\beta\,, \\ Q_\alpha^{(1)} &= \sqrt{\frac{1}{2}}\left[P_k (\sigma_k)_\alpha^{\ \beta} - ie(\bar\varphi\varphi - \bar\chi\chi)\delta_\alpha^\beta\right]\psi_\beta \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} H^{(0)} &= \pi_\varphi \pi_{\bar\varphi} + \pi_\chi \pi_{\bar\chi} + e^2(\bar\varphi\varphi + \bar\chi\chi)A_k^2 + eA_k(\bar\eta\sigma_k\eta - \bar\xi\sigma_k\xi)\,, \\ H^{(1)} &= e\sqrt{2}(\bar\psi\xi\bar\varphi + \bar\xi\psi\varphi) - e\sqrt{2}(\bar\psi\eta\bar\chi + \bar\eta\psi\chi)\,, \\ H^{(2)} &= \frac{1}{2}P_i P_i + \frac{1}{2}e^2(\bar\varphi\varphi - \bar\chi\chi)^2 \end{aligned} \tag{2.3}$$

($P_k = -i\partial/\partial A_k$, $\pi_\phi = -i\partial/\partial\phi$, etc). The different terms in eq. (2.1) are classified according to the powers of the small parameter $|\varphi|/|\mathbf{A}| \sim |\chi|/|\mathbf{A}|$. The leading order hamiltonian $H^{(0)}$ is quadratic with respect to the fast variables ϕ, χ and their superpartners ξ_α, η_α and represents a supersymmetric oscillator. The ground state of $H^{(0)}$ has zero energy and its wave function can be easily found:

$$\phi_0(\varphi, \bar\varphi, \chi, \bar\chi; \xi_\alpha, \eta_\alpha) = \frac{e}{\pi} A \exp\{-eA(\bar\varphi\varphi + \bar\chi\chi)\}\left[\xi^\alpha\eta_\alpha + \frac{\xi^\alpha(\sigma_k)_\alpha^{\ \beta}\eta_\beta A_k}{A}\right]. \tag{2.4}$$

Here A_k enter as parameters ($A = |\mathbf{A}|$). The characteristic values of φ, χ in the wave function (2.4) are $\varphi_{\rm char} \sim \chi_{\rm char} \sim 1/\sqrt{eA}$. We see that the assumption $A \gg |\varphi|, |\chi|$ is self-consistent when $\gamma = 1/(eA^3) \ll 1$. Now, γ is the actual Born-Oppenheimer parameter in $H^{\rm eff}$, the corrections over which we are going to find.

[3]The field ψ is defined here with the extra factor $-i$ compared to ref. [5].

To do this, we have to represent the total wave function of our system as a sum over the eigenfunctions of $H^{(0)}$:

$$\Psi(A_i, \psi_\alpha; \varphi, \bar{\varphi}, \chi, \bar{\chi}, \xi_\alpha, \eta_\alpha) = \sum_n r_n(A_i, \psi_\alpha)\phi_n(\varphi, \bar{\varphi}, \chi, \bar{\chi}; \xi_\alpha, \eta_\alpha)\,. \tag{2.5}$$

Then we write the Schrödinger equation $H\Psi = E\Psi$, assuming the energy E being small compared to the characteristic energies E_n of the excitations of $H^{(0)}$, express the coefficients $r_{n>0}$ via r_0 and cast the equation for r_0 thus obtained in the Schrödinger form. The operator acting on r_0 *is* called the effective hamiltonian.

It is technically convenient to calculate the effective supercharge rather than the effective hamiltonian. To leading order, the former is given by the matrix element[4]

$$Q_\alpha^{\text{eff}} = \langle Q_\alpha^{(1)} \rangle_{00} = \sqrt{\frac{1}{2}}(\sigma_k)_\alpha{}^\beta \psi_\beta P_k\,. \tag{2.6}$$

Taking the subleading correction into account, we obtain (see eqs. (15), (16) of ref. [5])

$$Q_\alpha^{\text{eff}} = \langle Q_\alpha^{(1)} \rangle_{00} - \sum_n{}' \frac{\langle Q_\alpha^{(1)} \rangle_{0n} \langle H^{(2)} \rangle_{n0}}{E_n} + \sum_{nm}{}' \frac{\langle Q_\alpha^{(1)} \rangle_{0n} \langle H^{(1)} \rangle_{nm} \langle H^{(1)} \rangle_{m0}}{E_n E_m}\,, \tag{2.7}$$

where the sums are done over the excited states of $H^{(0)}$. An explicit calculation described in appendix A gives us the first line in eq. (1.6) with

$$f(\mathbf{A}) = 1 - \frac{1}{4eA^3}\,. \tag{2.8}$$

The second line is obtained from the first by hermitean conjugation and the effective hamiltonian is calculated as the anticommutator $\frac{1}{2}\left\{\bar{Q}^\alpha, Q_\alpha\right\}_+$.

The bosonic part of the hamiltonian (1.6) describes the motion along a 3-dimensional manifold with the conformally flat metric $ds^2 = f^{-2}(\mathbf{A})d\mathbf{A}^2$. The whole hamiltonian represents a nonstandard $\mathcal{N} = 2$ supersymmetric extension of such σ-model (the standard one involves three complex fermionic variables ψ_i instead of two-component ψ_α and enjoys only $\mathcal{N} = 1$ supersymmetry). This nonstandard extension can be constructed only in the quantum mechanical limit and, in contrast to the standard one, does not allow for a field theory generalization.

The model (1.6) can be expressed in superfield (or better to say, supervariable) form [6]. Let us take a usual vector superfield $V(t, \theta_\alpha, \bar{\theta}^\alpha)$ involving photon and photino variables (spatial dependence is suppressed and no distinction between undotted and dotted indices is made). Introduce the real symmetric tensor supervariable

$$\Phi_{\alpha\beta} = (D_\alpha \bar{D}_\beta + D_\beta \bar{D}_\alpha)V\,, \tag{2.9}$$

where $D_\alpha, \bar{D}_\alpha$ are supersymmetric covariant derivatives. Remarkably, $\Phi_{\alpha\beta}$ is gauge invariant in the QM limit. The real supervariable $\Phi_k = -(1/4)\epsilon^{\beta\gamma}(\sigma_k)_\gamma{}^\alpha \Phi_{\alpha\beta}$ can be decomposed

[4]When deriving this, we used the fact that $\langle \bar{\varphi}\varphi \rangle_{00} = \langle \bar{\chi}\chi \rangle_{00}$ and also that the derivative of the wave function (2.4) with respect to A_i has no projection on $\phi_0(\varphi, \bar{\varphi}, \chi, \bar{\chi}; \xi_\alpha, \eta_\alpha)$.

into components as follows:

$$\begin{aligned}\Phi_k = &\ A_k + \bar{\tilde{\psi}}\sigma_k\theta + \bar\theta\sigma_k\tilde\psi + \epsilon_{kjp}\dot{A}_j\bar\theta\sigma_p\theta + D\bar\theta\sigma_k\theta + \\ &+ i(\bar\theta\sigma_k\dot{\tilde\psi} - \dot{\bar{\tilde\psi}}\sigma_k\theta)\bar\theta\theta + \frac{1}{4}\ddot{A}_k\theta^2\bar\theta^2\end{aligned} \tag{2.10}$$

(D is the auxilliary field). The real symmetric supervariable $\Phi_{\alpha\beta}$ satisfies the constraint

$$D_\alpha\Phi_{\beta\gamma} + D_\beta\Phi_{\alpha\gamma} + D_\gamma\Phi_{\alpha\beta} = 0 \tag{2.11}$$

and can be *defined* via this constraint.

Consider the action

$$S = \int dt d^2\theta d^2\bar\theta\ F(\mathbf{\Phi})\,, \tag{2.12}$$

where F is an arbitrary function. Expanding the action (2.12) into components, we obtain

$$\begin{aligned}\mathcal{L} = &\ \frac{1}{2f^2}\dot{A}_j\dot{A}_j + \frac{i}{2f^2}\left(\bar{\tilde\psi}\dot{\tilde\psi} - \dot{\bar{\tilde\psi}}\tilde\psi\right) - \frac{\partial_m f}{f^3}\epsilon_{mjp}\dot{A}_j\bar{\tilde\psi}\sigma_p\tilde\psi + \\ &+ \frac{D^2}{2f^2} + \frac{D\partial_k f}{f^3}\bar{\tilde\psi}\sigma_k\tilde\psi - \frac{1}{8}\partial^2\left(\frac{1}{f^2}\right)\bar{\tilde\psi}^2\tilde\psi^2\,,\end{aligned} \tag{2.13}$$

where we introduced

$$f(\mathbf{A}) = \sqrt{\frac{1}{2\partial^2 F(\mathbf{A})}}\,. \tag{2.14}$$

It is convenient to make the substitution $\tilde\psi = f(\mathbf{A})\psi$. Then $\bar\psi$ and ψ are canonically conjugated. After eliminating the auxilliary field D and deriving the hamiltonian by the standard rules, one can be easily convinced that it identically coincides with the hamiltonian (1.6). In our particular case,

$$F = \frac{\mathbf{\Phi}^2}{12} - \frac{1}{4e|\mathbf{\Phi}|}\ln|\mathbf{\Phi}|\,. \tag{2.15}$$

The model (2.12) can be easily generalized by introducing a set of r real supervariables $\mathbf{\Phi}^s$ satisfying the constraints (2.11). The action is written as

$$S = \int dt d^2\theta d^2\bar\theta F(\mathbf{\Phi}^s) \tag{2.16}$$

with arbitrary F. The bosonic part of (2.16) describes the motion on a $3r$-dimensional manifold with the metric

$$G_{is,jt} = 2\delta_{ij}\frac{\partial^2 F(\mathbf{A}^1,\ldots,\mathbf{A}^r)}{\partial A_l^s \partial A_l^t} \tag{2.17}$$

($i,j = 1,2,3$ and $s,t = 1,\ldots,r$).

The main result of this paper is that the effective action for the SYM quantum mechanics (1.1) with any gauge group can be cast in the form (2.16) with some particular function F depending on r (the rank of the group) supervariables $\mathbf{\Phi}^s$ described above.

JHEP04(2002)054

3. SU(2) theory

The simplest non-abelian model of the class (1.1) is based on the SU(2) gauge group. To find effective theory, we choose the $A_k^3 \equiv c_k$ and its superpartner $\lambda_\alpha^3 \equiv \psi_\alpha$ as the slow variables. The components $A_k^{1,2}$ will be treated as fast variables and integrated over. Heuristically, the variables $A_k^1 \pm iA_k^2$ play the same role as the charged scalar fields ϕ, χ in SQED. This is especially suggestive if the gauge

$$c_k A_k^{1,2} = 0 \tag{3.1}$$

is imposed — the number of remaining degrees of freedom would be just correct. In addition, if aligning c_k along the third axis and introducing

$$\begin{aligned} \phi &= \frac{1}{4}\left(A_1^1 - iA_1^2 + iA_2^1 + A_2^2\right) \\ \chi &= \frac{1}{4}\left(A_1^1 + iA_1^2 + iA_2^1 - A_2^2\right), \end{aligned} \tag{3.2}$$

the potential part of the hamiltonian has the same form as in SQED. The analogy should not, however, be pushed too far. Though the components $A_k^{1,2}$ and their superpartners play qualitatively the same role as the charged scalars and fermions in SQED, the total hamiltonian in the non-abelian case is different.

A direct calculation of the effective supercharges and hamiltonian along the same lines as it was done for SQED is not a simple task. The symmetry arguments dictate us, however, that the *functional form* of the effective hamiltonian is given, again, by eq. (1.6) (this is the only known $\mathcal{N} = 2$ supersymmetric structure involving three bosonic and two complex fermionic variables). The only question is what the function $f(\mathbf{c})$ is.

One thing can be said immediately: $f(\mathbf{c})$ should tend to 1 when $\mathbf{c}$ tends to infinity: in this limit the corrections in the Born-Oppenheimer expansion vanish. The parameter of the expansion is, again, $g|\mathbf{c}|^3$. If this parameter is large, the fluctuations of the fast variables $[A_i^{1,2}]^{\rm char} \sim 1/\sqrt{g|\mathbf{c}|}$ are small compared to $|\mathbf{c}|$. And if not, then not. This suggests that

$$f(\mathbf{c}) = 1 + \frac{C}{g|\mathbf{c}|^3} + \cdots, \tag{3.3}$$

which is similar to what we have found above for SQED, and only the coefficient C may be different.

To determine C, let us go back to the abelian case and consider a little bit more sophisticated problem of finding the effective hamiltonian not for the dimensionally reduced SQED (2.2), (2.3), but for the 4-dimensional field theory put in a small spatial box of length L. The system involves now an infinite number of dynamic variables — the Fourier harmonics of the fields $A_i(\mathbf{x})$, $\phi(\mathbf{x})$, $\xi(\mathbf{x})$, etc. However, the effective theory is written in terms of a finite number of slow variables, the zero Fourier harmonics $\mathbf{A} \equiv \mathbf{c}/L$ of the gauge fields and their superpartners. The effective quantum-mechanical hamiltonian has the same form as in eq. (1.6) (up to the dimensional factor $1/L$) with the function $f(\mathbf{c})$ given by the expression [5]

$$f^{\rm field\ \ theory}(\mathbf{c}) = 1 - \frac{e^2}{4}\sum_{\mathbf{n}} \frac{1}{|e\mathbf{c} - 2\pi\mathbf{n}|^3}. \tag{3.4}$$

The sum runs over all 3-dimensional integer $\mathbf{n}$. This sum diverges logarithmically at large $|\mathbf{n}|$. The origin of the divergence is very clear — it is just the charge renormalization. Substituting eq. (3.4) into eq. (2.13) (multiplied by L), keeping track of only logarithmically divergent part, and introducing the running effective charge

$$e^2(L) = e_0^2 \left[1 - \frac{e_0^2}{4\pi^2} \ln(\Lambda_{UV} L)\right], \tag{3.5}$$

we obtain the familiar renormalization of the kinetic term in the effective lagrangian,

$$\mathcal{L}^{\text{eff}} = \frac{L}{2} \frac{e_0^2}{e^2(L)} \dot{\mathbf{c}}\dot{\mathbf{c}} + \cdots \tag{3.6}$$

Multiplying $\mathcal{L}^{\text{eff}}$ by the Z factor, one can obtain the renormalized effective lagrangian with

$$f_{\text{reg}}(\mathbf{c}) = 1 - \frac{e^2}{4} \left[\sum_{\mathbf{n}} \frac{1}{|e\mathbf{c} - 2\pi\mathbf{n}|^3} - {\sum_{\mathbf{n}}}' \frac{1}{|2\pi\mathbf{n}|^3}\right]. \tag{3.7}$$

Note that the coefficient $1/4$ in eq. (2.8) *implies* the correct value $b_0 = -2$ of the one-loop coefficient of the β-function in SQED. If the same program is carried out for the SU(2) theory, the coefficient in eq. (3.6) should depend on the non-abelian effective charge with $b_0 = 3c_V = 6$. Therefore, the second term in eq. (3.4) acquires the factor -3 in the non-abelian case.[5] This fixes the coefficient $C = 3/4$ in eq. (3.3) and leads us to the result (1.7). This conclusion is confirmed by an explicit calculation of $f(\mathbf{c})$ in the Lagrange formalism [9].

4. Other gauge groups

Let us first remind some basic facts of the theory of Lie algebras (see e.g. the textbook [10]). The generators of an arbitrary Lie algebra $\mathfrak{g}$ form a linear space formed by r commuting generators of the Cartan subalgebra $\mathfrak{h}$, the positive root vectors $e_{\alpha_j} \equiv e_j$ and the negative root vectors $e_{-\alpha_j} \equiv f_j$. The relations

$$[h, e_j] = \alpha_j(h) e_j, \qquad [h, f_j] = -\alpha_j(h) f_j \tag{4.1}$$

hold, where $h \in \mathfrak{h}$ and $\alpha_j(h)$ are certain linear forms on the Cartan subalgebra called the (positive) *roots* of the Lie algebra $\mathfrak{g}$. The commutator $[e_j, f_j]$ is proportional to the *coroot* $\alpha_j^\vee$ lying in $\mathfrak{h}$. We can choose the normalization with $[e_j, f_j] = \alpha_j^\vee$.

Setting a natural metric on $\mathfrak{h}$ (with $\langle h, g\rangle = \text{Tr}\{hg\}$, $h, g \in \mathfrak{h}$) and the induced metric on the space of roots, one can define the matrix of the scalar products of the roots $c_{jj'} = \langle \alpha_j, \alpha_{j'}\rangle$. It is related to the so called *Cartan matrix*.[6] We normalize $d_j = c_{jj} = 1$ for the long roots. For Sp($2r$) and SO($2r+1$) there are also short roots with $d_j = 1/2$. The short root of G_2 has length $d_j = 1/3$.

[5] A similar sum, but without taking into account the dependence of the metric on $\mathbf{c}$ was written in recent [7]. See also [8] for a related analysis in pure Yang-Mills theory.

[6] The latter is usually defined as the matrix of scalar products of *simple* roots, whereas in our case α_j represent all positive roots. To be as clear and instructive as possible, we spell out in appendix B all the notations for the simplest nontrivial case of SU(3).

The coroots corresponding to the short roots are long, $\mathrm{Tr}\{\alpha_j^\vee \alpha_j^\vee\} \propto 1/d_j$. The following corollary is important for us. Consider the set of generators $S_j = (e_j, f_j, \alpha_j^\vee)$ realizing an embedding of su(2) into $\mathfrak{g}$. The Yang-Mills action $-\frac{1}{2g^2}\mathrm{Tr}\{F_{\mu\nu}F_{\mu\nu}\}$ restricted on the set S_j gives us the Yang-Mills action for the SU(2) group with the effective coupling constant

$$g^{(j)} = g\sqrt{d_j}\,. \tag{4.2}$$

We will also use the property

$$\sum_{j'} c_{jj'}\alpha_{j'}(h) = \frac{c_V}{2}\alpha_j(h)\,, \tag{4.3}$$

where c_V is the adjoint Casimir eigenvalue. A related property is

$$\sum_j \alpha_j(X^s)\alpha_j(Y^s) = \frac{c_V}{2}\sum_s X^s Y^s\,. \tag{4.4}$$

The effective hamiltonian of the theory (1.1) with an arbitrary gauge group G is expressed in terms of the slow variables A_k^s lying in $\mathfrak{h}$ and their superpartners.[7] The other components of the vector potential A_k^{+j} and A_k^{-j} directed along the root vectors e_j and f_j, respectively (for SU(2), $A_k^\pm = (A_k^1 \pm A_k^2)/2$), represent fast variables. We can impose the following gauge fixing condition,

$$A_k^{\pm j}\alpha_j\left(A_k^s\right) = 0\,, \tag{4.5}$$

where $\alpha_j\left(A_k^s\right)$ are the root linear forms of the arguments A_k^s. ($\alpha_j\left(A_k^s\right)$ is not just a number like $\alpha_j(h)$ is, but a 3-vector.) Eq. (4.5) involves $d-r$ gauge fixing conditions (d is the dimension of the group and r is its rank). After such a partial gauge fixing, the gauge group is broken down to its maximal torus $[\mathrm{U}(1)]^r$ and the problem is reduced to analyzing an abelian $[\mathrm{U}(1)]^r$ theory. Due to the conditions (4.5), A_k^{+j} and A_k^{-j} involve $2+2$ independent components. If aligning $\alpha_j\left(A_k^s\right)$ along the third spatial axis, these four components can be traded for two complex ones

$$\varphi^j = \frac{1}{2}\left(A_1^{-j} + iA_2^{-j}\right), \qquad \chi^j = \frac{1}{2}\left(A_1^{+j} + iA_2^{+j}\right). \tag{4.6}$$

The fields φ^j, χ^j are charged with respect to the set of r abelian gauge fields A_k^s. The same charges are carried by the fermion fields

$$\xi_\alpha^j = \sqrt{2}\lambda_\alpha^{-j}\,, \qquad \eta_\alpha^j = \sqrt{2}\lambda_\alpha^{+j}\,. \tag{4.7}$$

The quadratic in fast variables part of the hamiltonian is

$$\sum_j\left\{\pi_{\phi^j}\pi_{\bar\phi^j} + \pi_{\chi^j}\pi_{\bar\chi^j} + g^2(\bar\varphi^j\varphi^j + \bar\chi^j\chi^j)\left[\alpha_j(A_k^s)\right]^2 + g\alpha_j(A_k^s)(\bar\eta^j\sigma_k\eta^j - \bar\xi^j\sigma_k\xi^j)\right\}, \tag{4.8}$$

[7] An implicit assumption here is that classical vacua $F_{ij} = 0$ are given by constant gauge potentials lying in $\mathfrak{h}$. For orthogonal and exceptional groups, also nontrivial flat connections exist and are relevant [11]. This gives extra disconnected components in moduli space, but we will not consider this issue here.

which is rather analogous to eq. (2.3). Unfortunately, the full hamiltonian of the abelian theory obtained after a partial gauge fixing (4.5) is not simple. We are not going to tackle it explicitly, but rather reconstruct the effective hamiltonian from symmetry considerations, as we did for SU(2).

The effective hamiltonian enjoys $\mathcal{N}=2$ supersymmetry and involves $3r$ bosonic and $2r$ complex fermionic variables. In addition, it involves at most quadratic terms in momenta. The only known candidate theory has the form (2.16), and the only question is what is the function $F(\mathbf{\Phi}^s)$. To the leading Born-Oppenheimer order, it is quadratic in $\mathbf{\Phi}^s$,

$$F(\mathbf{\Phi}^s) = \frac{1}{12}\sum_s (\mathbf{\Phi}^s)^2 = \frac{1}{6c_V}\sum_j \left(\mathbf{\Phi}^{(j)}\right)^2 , \tag{4.9}$$

where we introduced the notation $R^{(j)} = \alpha_j(R^s)$ and used the relation (4.4). Integrating (4.9) over $d^4\theta$, we obtain

$$\mathcal{L}^{\rm eff} = \frac{1}{2}\sum_s \left[\dot{A}_k^s \dot{A}_k^s + i\left(\bar{\psi}^s\dot{\psi}^s - \dot{\bar{\psi}}^s\psi^s\right)\right] \tag{4.10}$$

in accordance with eq. (1.4).

We are set to calculate the leading nontrivial Born-Oppenheimer corrections to $\mathcal{L}^{\rm eff}$. Born-Oppenheimer approximation is valid when the fluctuations of the fast variables φ^j, χ^j are much less than the characteristic values of $|\mathbf{A}^s|$. It is clear from eq. (4.8) that $(\varphi^j)^{\rm char}$, $(\chi^j)^{\rm char}$ are of order $1/g\sqrt{|\mathbf{A}^{(j)}|}$, which should be compared with $|\mathbf{A}^{(j)}|$. Thereby the Born-Oppenheimer expansion makes sense when *all* the parameters

$$\gamma^j = \frac{1}{g|\mathbf{A}^{(j)}|^3}$$

are small. The leading corrections to $\mathcal{L}^{\rm eff}$ and, in particular, to the metric should be linear in γ^j. This means that the leading nontrivial correction to the leading order result (4.9) for the function $F(\mathbf{\Phi}^s)$ represents a linear form of $\delta^j = (1/|\mathbf{\Phi}^{(j)}|)\ln|\mathbf{\Phi}^{(j)}|$ (cf. eq. (2.15)). In other words,

$$F(\mathbf{\Phi}^s) = \frac{1}{6c_V}\sum_j \left[\left(\mathbf{\Phi}^{(j)}\right)^2 - \frac{C_j}{|\mathbf{\Phi}^{(j)}|}\ln|\mathbf{\Phi}^{(j)}|\right] , \tag{4.11}$$

where C_j are some numerical coefficients, to be determined. Eq. (2.17) provides us with the metric

$$G_{kp,st} = \delta_{kp}\frac{1}{c_V}\sum_j \alpha_j^s\alpha_j^t\left[1 - \frac{C_j c_V}{g|\mathbf{A}^{(j)}|^3}\right] . \tag{4.12}$$

This gives us the kinetic part in the effective lagrangian

$$\mathcal{L}^{\rm eff}_{\rm kin} = \frac{1}{c_V}\sum_j \dot{\mathbf{A}}^{(j)}\dot{\mathbf{A}}^{(j)}\left[1 - \frac{C_j c_V}{g|\mathbf{A}^{(j)}|^3}\right] . \tag{4.13}$$

Let us consider the situation when one of the root forms $|\mathbf{A}^{(j_0)}|$ is much smaller than all others (and the corresponding parameter γ^{j_0} is much bigger than all others, but still small). Then one can neglect all the corrections except the one with $j = j_0$ and write

$$\mathcal{L}^{\text{eff}}_{\text{kin}} = \frac{1}{c_V}\left\{\dot{\mathbf{A}}^{(j_0)}\dot{\mathbf{A}}^{(j_0)}\left[1 - \frac{C_{j_0}c_V}{g|\mathbf{A}^{(j_0)}|^3}\right] + \sum_{j\neq j_0}\dot{\mathbf{A}}^{(j)}\dot{\mathbf{A}}^{(j)}\right\}. \tag{4.14}$$

The relevant dynamics is determined by the variables $\mathbf{A}^{(j_0)}$ and their superpartners. It would not be correct, however, just to cross out the terms with $j \neq j_0$. The variables $\mathbf{A}^{(j\neq j_0)}$ have nonzero projections on $\mathbf{A}^{(j_0)}$:

$$\mathbf{A}^{(j)} = \frac{c_{jj_0}}{d_{j_0}}\mathbf{A}^{(j_0)} + \text{orthogonal combinations}\,.$$

Substituting this in eq. (4.13) and using the property

$$\sum_j c^2_{jj_0} = d_{j_0}\frac{c_V}{2}\,,$$

which is a corollary of eq. (4.4), we obtain

$$\mathcal{L}^{\text{eff}}_{\text{kin}} \approx \frac{1}{2d_{j_0}}\dot{\mathbf{A}}^{(j_0)}\dot{\mathbf{A}}^{(j_0)}\left[1 - \frac{2d_{j_0}C_{j_0}}{g|\mathbf{A}^{(j_0)}|^3}\right] + \text{orthogonal terms}\,. \tag{4.15}$$

For long roots, $d_{j_0} = 1$ and we can directly compare this with the effective lagrangian of the SU(2) theory (see eq. (1.7)). We obtain

$$C_j = \frac{3}{4}\,. \tag{4.16}$$

For short roots, we have first to renormalize $\mathbf{A}^{(j_0)} \to \sqrt{d_{j_0}}\mathbf{A}^{(j_0)}$ to bring the kinetic term into the standard form. Identifying the action (4.15) with that of the corresponding SU(2) theory and bearing in mind that the SU(2) coupling constant is related to g according to (4.2), we derive that the relation (4.16) holds also for the short roots.

Substituting eq. (4.16) into the lagrangian (4.13), calculating the canonical momenta, and evaluating the supercharges with the Nöther theorem, we obtain

$$Q^{\text{eff}}_\alpha = \frac{\sqrt{2}}{c_V}\sum_j\left\{(\sigma_k)_\alpha{}^\beta\psi^{(j)}_\beta P^{(j)}_k\left[1 + \frac{3c_V}{8g|\mathbf{A}^{(j)}|^3}\right] - \right.$$
$$\left. - \frac{9ic_V A^{(j)}_k}{8g|\mathbf{A}^{(j)}|^5}\bar{\psi}^{(j)}\sigma_k\psi^{(j)}\psi^{(j)}_\alpha\right\} + \cdots\,, \tag{4.17}$$

where the dots stand for the terms of higher order in γ^j.[8] The effective hamiltonian is

$$H^{\text{eff}} = \frac{1}{c_V}\sum_j\left\{\left[\mathbf{P}^{(j)}\right]^2\left[1 + \frac{3c_V}{4g|\mathbf{A}^{(j)}|^3}\right] - \frac{9c_V A^{(j)}_p}{8g|\mathbf{A}^{(j)}|^5}\epsilon_{lkp}\bar{\psi}^{(j)}\sigma_l\psi^{(j)}P^{(j)}_k - \right.$$
$$\left. - \frac{9c_V}{8g|\mathbf{A}^{(j)}|^5}\left(\bar{\psi}^{(j)}\psi^{(j)}\right)^2\right\} + \cdots\,. \tag{4.18}$$

[8]These terms are important to close the algebra. If only the terms explicitly displayed in eq. (4.17) were taken into account, the property $\{Q_\alpha, Q_\beta\}_+ = 0$ would not hold.

JHEP04(2002)054

If the original non-abelian theory is placed in a finite spatial box, one has also to take into account the higher Fourier harmonics of the charged fields (4.6), (4.7) and of the ghosts. After this, the factor multiplying the derivative term in eq. (4.13) is traded for the expression

$$1 - \frac{3c_V}{4g} \sum_{\mathbf{n}} \frac{1}{|\mathbf{A}^{(j)} - 2\pi\mathbf{n}/g|^3}\,.$$

The divergent part of this sum gives the renormalization of the field theory coupling constant with the correct coefficient.

Acknowledgments

I am deeply indebted to E. Akhmedov, M. Shifman, A. Vainshtein, and P. van Baal for illuminating discussions. I appreciate warm hospitality extended to me at the University of Minnesota, where this work was mostly done.

A. Calculation of the effective supercharge for SQED

The calculation of the effective supercharges and hamiltonian for SQED was done in ref. [5] but was not presented there in enough details, so that it was not absolutely trivial even for myself to reproduce it after 15 years. We decided to redo the calculations in the most explicit way.

To calculate the matrix elements in eq. (2.7), we need to know not only the ground state (2.4) of the hamiltonian $H^{(0)}$, but also the relevant excited states. The wave functions of the states providing for nonzero matrix elements $\langle H^{(2)} \rangle_{n0}$ were written in ref. [5]:

$$\begin{aligned}
|+\rangle &= \sqrt{2}\frac{e}{\pi} A[1 - eA(\bar\phi\phi + \bar\chi\chi)] \times \\
&\quad \times \exp\{-eA(\bar\varphi\varphi + \bar\chi\chi)\} \left[\xi^\alpha \eta_\alpha + \frac{\xi^\alpha (\sigma_k)_\alpha{}^\beta \eta_\beta A_k}{A}\right], \\
|-\rangle &= \sqrt{2}\frac{e}{\pi} A^2 (\bar\phi\phi - \bar\chi\chi) \exp\{-eA(\bar\varphi\varphi + \bar\chi\chi)\} \left[\xi^\alpha \eta_\alpha + \frac{\xi^\alpha (\sigma_k)_\alpha{}^\beta \eta_\beta A_k}{A}\right], \\
|\mathbf{l}\rangle &= \sqrt{2} \cdot \frac{e}{\pi} A \exp\{-eA(\bar\varphi\varphi + \bar\chi\chi)\} \xi^\alpha (\sigma_k)_\alpha{}^\beta \eta_\beta \left(\delta_{lk} - \frac{A_l A_k}{A^2}\right).
\end{aligned} \tag{A.1}$$

They are all bosonic and have energy $2eA$. The states $|\mathbf{l}\rangle$ are not all linearly independent and normalized according to

$$\langle \mathbf{l} | \mathbf{k} \rangle = \delta_{lk} - \frac{A_l A_k}{A^2}\,. \tag{A.2}$$

We can evaluate now the second term in eq. (2.7). Note first of all that the wave function $(e^2/2)(\bar\varphi\varphi - \bar\chi\chi)^2|0\rangle$ has the projection on the vacuum state (irrelevant in the context of eq. (2.7)) and also the projection on the state $|+\rangle$,

$$\langle + | \, \frac{e^2}{2} \, (\bar\varphi\varphi - \bar\chi\chi)^2 \, |\mathbf{0}\rangle = -\frac{1}{2\sqrt{2}A^2}\,, \tag{A.3}$$

while all other matrix elements are zero. The result of the action of the laplacian Δ_A on the vacuum state is not just a function, but a differential operator,

$$\Delta_A|\mathbf{0}\rangle = |\mathbf{0}\rangle\Delta_A + \sqrt{2}\left(|+\rangle\frac{A_i}{A^2} + |\mathbf{l}\rangle\frac{1}{A}\right)\frac{\partial}{\partial A_l} + |+\rangle\frac{1}{\sqrt{2}A^2}\,. \tag{A.4}$$

Combining eq. (A.3) and eq. (A.4), we arrive at the following expressions for nondiagonal matrix elements

$$\begin{aligned} \langle +|\, H^{(2)}\,|\mathbf{0}\rangle &= -\frac{1}{\sqrt{2}A^2}\left(1 + A_k\frac{\partial}{\partial A_k}\right), \\ \langle \mathbf{l}|\, H^{(2)}\,|\mathbf{0}\rangle &= -\frac{1}{\sqrt{2}A}\frac{\partial}{\partial A_l}\,. \end{aligned} \tag{A.5}$$

We need also the matrix elements of the supercharge $Q^{(1)}_\alpha$. They are

$$\begin{aligned} \langle \mathbf{0}|\, Q^{(1)}_\alpha\,|+\rangle &= \frac{iA_k}{2A^2}(\sigma_k)_\alpha^{\ \beta}\psi_\beta\,, \\ \langle \mathbf{0}|\, Q^{(1)}_\alpha\,|\mathbf{l}\rangle &= \frac{i}{2A}(\sigma_k)_\alpha^{\ \beta}\psi_\beta\left(\delta_{lk} - \frac{A_lA_k}{A^2}\right), \\ \langle \mathbf{0}|\, Q^{(1)}_\alpha\,|-\rangle &= -\frac{i}{2A}\psi_\alpha\,. \end{aligned} \tag{A.6}$$

To evaluate the middle term in eq. (2.7) we need only two first matrix elements. The corresponding correction to the effective supercharge is

$$\delta_{H_2}Q^{\rm eff}_\alpha = \frac{i}{4\sqrt{2}eA^3}(\sigma_k)_\alpha^{\ \beta}\psi_\beta\left(\frac{\partial}{\partial A_k} + \frac{A_k}{A^2}\right). \tag{A.7}$$

Let us discuss now the last term in eq. (2.7). Besides the bosonic intermediate states (A.1), there are also fermionic states providing for nonzero matrix elements $\langle H^{(1)}\rangle_{m0}$. One of such states has the eigenfunction

$$\phi^\gamma_\xi = 2\frac{e}{\pi}A^{3/2}\chi\exp\{-eA(\bar\varphi\varphi + \bar\chi\chi)\}\left[\xi^\gamma + \xi^\alpha(\sigma_k)_\alpha^{\ \gamma}\frac{A_k}{A}\right] \tag{A.8}$$

and there are three others similar. The important fact is that all these states have the same energy $E_{\rm ferm} = 2eA$. And this means that we can do the sum $\sum_m$ in eq. (2.7) without tears and write

$$\sum_m{}' \langle H^{(1)}\rangle_{nm}\langle H^{(1)}\rangle_{m0}\frac{1}{E_m} = \frac{1}{2eA}\langle [H^{(1)}]^2\rangle_{n0}\,. \tag{A.9}$$

The explicit calculation gives

$$\begin{aligned} \langle +|\,[H^{(1)}]^2\,|\mathbf{0}\rangle &= -\frac{e\sqrt{2}}{A}\,, \\ \langle -|\,[H^{(1)}]^2\,|\mathbf{0}\rangle &= -\frac{e\sqrt{2}A_i}{A^2}\bar\psi\sigma_i\psi\,, \\ \langle \mathbf{l}|\,[H^{(1)}]^2\,|\mathbf{0}\rangle &= \frac{ie\sqrt{2}A_k}{A^2}\epsilon_{plk}\bar\psi\sigma_p\psi\,. \end{aligned} \tag{A.10}$$

The corresponding correction to Q_α^{eff} is

$$\delta_{H_1} Q_\alpha^{\text{eff}} = \frac{iA_k}{4\sqrt{2}eA^5}\left(3\psi_\alpha\bar{\psi}\sigma_k\psi - (\sigma_k)_\alpha{}^\beta\psi_\beta\right). \tag{A.11}$$

We obtain finally

$$Q_\alpha^{\text{eff}} = \frac{1}{\sqrt{2}}(\sigma_k)_\alpha{}^\beta\psi_\beta\left(1-\frac{1}{4eA^3}\right)P_k + \frac{3iA_k}{4\sqrt{2}eA^5}\psi_\alpha\bar{\psi}\sigma_k\psi\,. \tag{A.12}$$

This operator acts on the coefficient $r_0(\mathbf{A},\psi_\alpha)$ in the expansion (2.5). As was explained in ref. [5], the standard normalization condition for the total wave function bring about the extra factor $f^{-2}(\mathbf{A})$, $f(\mathbf{A}) = 1-1/(4eA^3)$, in the normalization condition for $r_0(\mathbf{A},\psi_\alpha)$. The effective supercharge acting on the effective wave function with the standard normalization $\Psi^{\text{eff}}(\mathbf{A},\psi_\alpha) = f^{-1}(\mathbf{A})r_0(\mathbf{A},\psi_\alpha)$, is obtained by wrapping

$$Q_\alpha^{\text{eff}} \longrightarrow f^{-1}(\mathbf{A})Q_\alpha^{\text{eff}}f(\mathbf{A})$$

and we arrive at the expression which coincides, indeed, with the first line of eq. (1.6).

B. Roots and root vectors for SU(3)

The group SU(3) has rank 2, there are two commuting generators $\lambda^3/2$ and $\lambda^8/2$ (λ^a are the standard Gell-Mann matrices). There are 3 positive roots. The root vectors are

$$\begin{aligned} e_1 &= \frac{1}{2}(\lambda^1+i\lambda^2)\,, & e_2 &= \frac{1}{2}(\lambda^6+i\lambda^7)\,, & e_3 &= \frac{1}{2}(\lambda^4+i\lambda^5)\,,\\ f_1 &= \frac{1}{2}(\lambda^1-i\lambda^2)\,, & f_2 &= \frac{1}{2}(\lambda^6-i\lambda^7)\,, & f_3 &= \frac{1}{2}(\lambda^4-i\lambda^5)\,. \end{aligned} \tag{B.1}$$

The relevant root forms are

$$\alpha_1(A_k^s) = A_k^3\,, \qquad \alpha_2(A_k^s) = \frac{-A_k^3+\sqrt{3}A_k^8}{2}\,, \qquad \alpha_3(A_k^s) = \frac{A_k^3+\sqrt{3}A_k^8}{2} \tag{B.2}$$

and similarly for $\alpha_j(\lambda_\alpha^s)$. One can observe that $\alpha_3 = \alpha_1+\alpha_2$. The matrix of scalar products $c_{jj'}$ is

$$c_{jj'} = \frac{1}{2}\begin{pmatrix} 2 & -1 & 1\\ -1 & 2 & 1\\ 1 & 1 & 2\end{pmatrix}. \tag{B.3}$$

It has an extra factor 1/2 compared to a more usual definition [12]. With eqs. (B.2) and (B.3) in hand, one can explicitly check that the relations (4.4), (4.3) hold.

References

[1] E. Witten, *Constraints on supersymmetry breaking*, *Nucl. Phys.* **B 202** (1982) 253.

[2] A.V. Smilga, *Super–Yang-Mills quantum mechanics and supermembrane spectrum*, in *Proc. Int. Workshop on Supermembranes and Physics in* $2+1$ *dimensions*, Trieste, July 1989, M.J. Duff, C.N. Pope, and E. Sezgin eds., Worlds Scientific, Singapore 1987.

[3] B. de Witt, M. Lüscher and H. Nicolai, *The supermembrane is unstable*, *Nucl. Phys.* **B 320** (1989) 135.

[4] V.G. Kac and A.V. Smilga, *Normalized vacuum states in $N = 4$ supersymmetric Yang-Mills quantum mechanics with any gauge group*, *Nucl. Phys.* **B 571** (2000) 515 [hep-th/9908096].

[5] A.V. Smilga, *Perturbative corrections to effective zero mode hamiltonian in supersymmetric QED*, *Nucl. Phys.* **B 291** (1987) 241.

[6] E.A. Ivanov and A.V. Smilga, *Supersymmetric gauge quantum mechanics: superfield description*, *Phys. Lett.* **B 257** (1991) 79.

[7] P. van Baal, *The Witten index beyond the adiabatic approximation*, hep-th/0112072.

[8] M. Lüscher, *Some analytic results concerning the mass spectrum of Yang-Mills gauge theories on a torus*, *Nucl. Phys.* **B 219** (1983) 233.

[9] E.T. Akhmedov and A.V. Smilga, *On the relation between effective supersymmetric actions in different dimensions*, hep-th/0202027.

[10] D.P. Ẑelobenko, *Compact Lie groups and their representations*, American Mathematical Society, Providence 1973.

[11] E. Witten, *Toroidal compactification without vector structure*, *J. High Energy Phys.* **02** (1998) 006 [hep-th/9712028];
A. Keurentjes, A. Rosly and A.V. Smilga, *Isolated vacua in supersymmetric Yang-Mills theories*, *Phys. Rev.* **D 58** (1998) 081701 [hep-th/9805183];
V.G. Kac and A.V. Smilga, *Vacuum structure in supersymmetric Yang-Mills theories with any gauge group* in *The many faces of the superworld*, M.A. Shifman ed., World Scientific, 2000 [hep-th/9902029];
A. Keurentjes, *Nontrivial flat connections on the 3 torus, 1. G_2 and the orthogonal groups*, *J. High Energy Phys.* **05** (1999) 001 [hep-th/9901154]; *Nontrivial flat connections on the three torus, 2. The exceptional groups F_4 and $E_{6,7,8}$*, *J. High Energy Phys.* **05** (1999) 014 [hep-th/9902186].

[12] R. Slansky, *Group theory for unified model building*, *Phys. Rept.* **79** (1981) 1.